UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH7402**

ASSESSMENT PATTERN : **MATH7402A**

MODULE NAME : **Mathematical Methods 4**

DATE : **18-May-10**

TIME : **10:00**

TIME ALLOWED : **2 Hours 0 Minutes**

2009/10-MATH7402A-001-EXAM-118

TURN OVER

All questions may be attempted but only marks obtained on the best four *solutions will count.*
The use of an electronic calculator is not *permitted in this examination.*

(1) The temperature distribution $u(x,t)$ in a rod of unit length and unit thermal diffusivity evolves according to the heat equation

$$u_t = u_{xx}, \qquad t > 0, \quad 0 < x < 1.$$

One end of the rod is perfectly insulated while the other end radiates heat, giving boundary conditions

$$\begin{aligned} u_x(0,t) &= 0, \\ u_x(1,t) &= -\alpha u(1,t), \end{aligned}$$

where α is a positive constant. The temperature in the rod is initially uniform, $u(x,0) = T_0$.

Show that the temperature in the interior of the rod for $t \geq 0$ is given by

$$u(x,t) = T_0 \sum_{n=0}^{\infty} \frac{4 \sin p_n}{2 + \sin 2p_n} \cos p_n x \, e^{-p_n^2 t},$$

where the $\{p_n\}$ are a sequence of constants given by the roots of the equation

$$\tan p = \frac{\alpha}{p}.$$

Illustrate graphically that there are infinitely many such roots.

[You may use the result that, for a Sturm-Liouville system defined on the interval $0 \leq x \leq 1$, any function $y(x)$ on that interval can be expressed as a generalised Fourier series in the eigenfunctions $X_n(x)$ of the Sturm-Liouville system as

$$y(x) = \sum_{n=1}^{\infty} \frac{(y, X_n)}{(X_n, X_n)} X_n(x)$$

where the inner product (f, g) should be defined for any two functions $f(x)$ and $g(x)$.]

(2) Show that the differential equation

$$(x^4+x^2)\frac{d^2y}{dx^2}+3(x^3-x)\frac{dy}{dx}+4y=0,$$

has a regular singular point at $x=0$.

Show that one solution to the equation has the form

$$y_1(x)=\sum_{n=0}^{\infty} b_n x^{2n+\lambda},$$

where λ takes a value to be determined, $b_0=1$ and for $n\geq 1$

$$b_n=(-1)^n\frac{2n+\lambda}{\lambda}.$$

Find a second independent solution $y_2(x)$ of the equation.

(3) (a) Axisymmetric solutions $u(r,\theta)$ of Laplace's equation in spherical geometry satisfy

$$\nabla^2 u=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)=0.$$

Using the method of separation of variables, show that the general solution of Laplace's equation that is regular at the poles ($\theta=0,\pi$) can be written

$$u(r,\theta)=\sum_{n=0}^{\infty}\left(A_n r^n+\frac{B_n}{r^{n+1}}\right)P_n(\cos\theta).$$

[You may assume that the only solutions of $(1-z^2)w''-2zw'+\alpha(\alpha+1)w=0$ which are regular at $z=\pm 1$ occur for $\alpha=n$ (n integer) and are the Legendre polynomials $P_n(z)$.]

(b) Find the steady temperature distribution everywhere within a perfectly conducting metal sphere of unit radius when a temperature distribution

$$u(1,\theta)=T_0\cos 2\theta$$

is applied to its surface.

[Hint: Rodrigues' formula

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n,$$

can be used to calculate the first few Legendre polynomials.]

(4) (a) You are given that solutions of Bessel's equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(\lambda^2x^2-n^2\right)y=0 \qquad (n \text{ integer}),$$

which are regular at $x=0$ are the Bessel functions of the first kind $y(x) = J_n(\lambda x)$. For fixed n there exists an infinite sequence $\{\lambda_i\}$ satisfying $J_n(\lambda_i)=0$. Show that

$$\int_0^1 xJ_n(\lambda x)J_n(\lambda_i x)\,dx = \frac{\lambda_i J_n(\lambda)J_n'(\lambda_i)}{\lambda^2-\lambda_i^2}.$$

Hence, considering both $i=j$ and $i\neq j$, find

$$\int_0^1 xJ_n(\lambda_j x)J_n(\lambda_i x)\,dx.$$

(b) Many properties of the Bessel functions can be obtained from the *generating function*

$$G(x,t)=\exp\left(\frac{x}{2}\left(t-\frac{1}{t}\right)\right)=\sum_{n=-\infty}^{\infty}t^nJ_n(x).$$

Show that

$$\begin{aligned}
&\text{(i)} & J_n'(x) &= \frac{1}{2}\left(J_{n-1}(x)-J_{n+1}(x)\right)\\
&\text{(ii)} & 2nJ_n(x) &= x\left(J_{n-1}(x)+J_{n+1}(x)\right).
\end{aligned}$$

(5) A real function $f(x)$ and its Fourier transform $\hat{f}(k)$ are related through

$$\hat{f}(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-ikx}\,dx \quad \text{and} \quad f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(k)e^{ikx}\,dk.$$

(a) Find the Fourier transform of

$$f(x)=\frac{1}{1+x^2}.$$

(b) Use Fourier transforms to show that the solution of the heat equation problem

$$u_t=u_{xx}, \qquad t\geq 0, \quad -\infty<x<\infty.$$

subject to the initial condition $u(x,0)=f(x)$ is given by

$$u(x,t)=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}f(y)\exp\left(-\frac{(x-y)^2}{4t}\right)dy.$$

[You may quote the result $\int_{-\infty}^{\infty}\cos kx\ \exp\left(-\frac{x^2}{a^2}\right)dx = a\sqrt{\pi}\exp\left(-\frac{k^2a^2}{4}\right)$, but must prove all other results that you use.]

(6) A function $f(t)$ defined on $[0, \infty)$ and its Laplace transform $\bar{f}(s)$ are related by

$$\bar{f}(s) = \int_0^{\infty} f(t)e^{-st}\,dt, \qquad f(t) = \frac{1}{2\pi i}\int_{\Gamma} \bar{f}(s)e^{st}\,ds.$$

(a) Use the forward transform to:

(i) Calculate the Laplace transform of e^{-at}, where a is a real constant.

(ii) Obtain a formula for the Laplace transform of $\dot{f}(t)$ (where the dot denotes the time derivative d/dt).

(iii) Calculate the Laplace transform of the convolution

$$(f * g)(t) = \int_0^t f(t-u)g(u)\,du.$$

(b) Sketch on the complex plane the path Γ for the integral in the inverse transform formula. Comment on the value(s) taken by any constants introduced.

(c) A coupled system $(x(t), y(t))$, subject to initial conditions $x(0) = y(0) = 0$, evolves according to

$$\begin{aligned} \dot{x}(t) + 2x(t) + y(t) &= F(t) \\ \dot{y}(t) + y(t) + 6x(t) &= 0 \end{aligned}$$

where $F(t)$ is a prescribed forcing function. Using Laplace transforms, or otherwise, show that

$$x(t) = \frac{1}{5}\int_0^t F(t-u)\left(2e^{u} + 3e^{-4u}\right)\,du.$$